Excluding cycles with a fixed number of chords

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Abstract

Trotignon and Vušković completely characterized graphs that do not contain cycles with exactly one chord. In particular, they show that such a graph G has chromatic number at most $\max(3, \omega(G))$. We generalize this result to the class of graphs that do not contain cycles with exactly two chords and the class of graphs that do not contain cycles with exactly three chords.

More precisely we prove that graphs with no cycle with exactly two chords have chromatic number at most 6. And a graph G with no cycle with exactly three chords have chromatic number at most $\max(96, \omega(G) + 1)$.

1 Introduction

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors needed to vertex-color G such that two adjacent vertices receive distinct colors. A clique is a graph such that every two vertices are adjacent. The *clique number* of a graph G, denoted by $\omega(G)$, is the number of vertices of the largest clique in G. A class of graphs is *hereditary* if, for any graph G in the class, every subgraph of G is in the class. We say that a graph G is H*-free* if G does not contain the graph H as an induced subgraph. If $\mathcal{H}$ is a class of graphs we say that a graph G is $\mathcal{H}$-free if for any $H \in \mathcal{H}$, G is H-free. Classes of graphs defined by forbidding some graphs as induced subgraphs are clearly hereditary.

It is clear that $\omega(G)$ is a lower bound of $\chi(G)$ since vertices of a clique are colored with pairwise distinct colors. Gyárfás introduced the following

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notion [6]: a graph is said to be χ-*bounded* if there exists a function f such that for every subgraph H of G, $\chi(H) \leq f(\omega(H))$. A class of graphs $\mathcal{C}$ is said to be χ-*bounded* is every graph in the class is χ-bounded. Observe that, in order to prove that a hereditary class $\mathcal{C}$ is χ-bounded by a function f, it is enough to prove that for any graph G in $\mathcal{C}$, $\chi(G) \leq f(\omega(G))$. For instance, graphs χ-bounded by the function $f(x) = x$ are known as *perfect graphs*. M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas proved [4] that perfect graphs are exactly the graphs that do not admit odd cycles of length at least 5 nor complement of odd cycles of length at least 5, solving the famous strong perfect graph conjecture proposed by C. Berge [2]. So, a natural question arises:

Question 1.1 What kind of induced structure is needed to be forbidden in order to get a χ-bounded class?

Let us now survey some results on χ-boundedness by emphasizing on what different meanings "structure" can take.

If H is a graph, we denote by *Forb(H)* the class of H-free graphs. A first way to tackle the problem is to determine for which graphs H, Forb(H) is χ-bounded. For example, it is proved in [6] that Forb(P_k) is χ-bounded (where P_k denotes the chordless path of length k). In [7], Erdős proved that there exists graphs with arbitrarily large chromatic number and arbitrarily large girth. So, if H contains a cycle, Forb(H) is not χ-bounded. It is actually conjectured in [6] that Forb(H) is χ-bounded if and only if H is a forest. The deeper results concerning this conjecture are certainly results of Kierstead and Penrice [8] and Kierstead and Zhu [9] proving that the conjecture holds for every tree of radius at most 2 and several trees of radius 3. To get out from this conjecture, we need to forbid a class of graph $\mathcal{H}$ such that $\mathcal{H}$ contains graphs with arbitrarily large girth.

A second way to forbid induced structures is the following: fix a graph H, and forbid every induced subdivision of H. We denote by *Forb*(H)* the class of graphs that do not contain induced subdivisions of H. The class Forb*(H) has been proved to be χ-bounded for a number of examples. The most beautiful one is certainly the proof by Scott [12] that for any forest F, Forb*(F) is χ-bounded. In the same paper he conjectured that, for any graph H, Forb*(H) is χ-bounded. Unfortunately, this conjecture has recently been disproved by Kozik *et al.* [10]. Based on this work, Chalopin *et al.* [3] gave a precise description of a number of graphs H for which Forb*(H) is not χ-bounded. There is no general conjecture on which H has to be forbidden in order to ensure Forb*(H) is χ-bounded.

A third way is to forbid a graph H for which some edges can be subdivided but some cannot. More generally, to forbid a class of graphs $\mathcal{H}$ such that, for each $H \in \mathcal{H}$, some edges can be subdivided and some cannot. A few classes defined this way has been studied (see [1] and [11] for instance). In [11], Trotignon and Vušković proved that the class of graphs that do not contain cycles with a unique chord is χ-bounded by the function $max(3, \omega(G))$. Forbidding cycles with a unique chord is equivalent to forbid a diamond (a diamond is a cycle of length 4 with a diagonal) such that every edge but the diagonal can be subdivided.

A *k-cycle* is a chordless cycle with exactly k chords. We call $\mathcal{C}_k$ the class of k-cycle-free graphs i.e. the class of graphs that do not contain cycles with exactly k chords. So, the cited result on the class of graphs that do not contain a cycle with a unique chord may be rephrased as follows : $\mathcal{C}_1$ is χ-bounded.

The two main results of this paper are that both $\mathcal{C}_2$ (see Theorem 4.1) and $\mathcal{C}_3$ (see Theorem 5.3) are χ-bounded. The statement of Theorem 4.1 deals with a super-class of $\mathcal{C}_2$, see Section 4 for more details. Since graphs which do not contain a 2-cycle do not contain K_4 as an induced subgraph, proving χ-boundedness is equivalent to prove that the chromatic number is bounded. We actually prove that the chromatic number in this case is at most 6. An immediate lower bound on the chromatic number is 3, given by odd cycles. Close the gap between these two values can be an interesting point.

The class $\mathcal{C}_3$, contrary to $\mathcal{C}_2$ that does not admit graphs with cliques larger than the triangle (because K_4 is a 2-cycle), admits graphs containing arbitrarily large cliques. We prove that the chromatic number of a graph $G \in \mathcal{C}_3$ is at most $\max(96, \omega(G) + 1)$. In addition we provide examples of graphs G with arbitrarily large clique such that $\chi(G) = \omega(G) + 1$, showing that our bound is asymptotically tight. Nevertheless, lower bound of 96 is surely far away from an optimal bound for graphs in $\mathcal{C}_3$ that do not contain large cliques.

Here is an outline of the paper. In Section 2, we give some terminologies and in Section 3 we describe the general method used in the proofs. Section 4 is concerned with the class $\mathcal{C}_2$ and Section 5 is concerned with the class $\mathcal{C}_3$. We also propose the following conjecture suggested by our results:

Conjecture 1.2 *For any integer $k \geq 4$, $\mathcal{C}_k$ is χ-bounded.*

2 Terminologies and notations

For standard definition on graphs, the reader should refer to classical books of graph theory, such as [5]. Let G be a graph, x a vertex of G and S a subset of vertices of G. We denote by $N(x)$ the set of neighbors of x, by $N_S(x)$ the set of neighbors of x in S, and by $N(S)$ the set of vertices of $V(G) \setminus S$ that have a neighbor in S. We denote by $d(x)$ the degree of x and by $d_S(x)$ the degree of x in S, i.e. the number of neighbors of x in S. We denote by $G[S]$ the subgraph of G induced by S, and $G \setminus S$ denotes $G[V(G) \setminus S]$. S is a *cutset* of G if $G \setminus S$ has more than one connected component. If S induces a clique, then S is a *clique cutset*. If $\{x\}$ is a cutset of G, then x is a *cutvertex*. Note that a cutvertex is a clique cutset.

A *path* P is a sequence of distinct vertices $p_1p_2 \dots p_k$, $k \geq 1$, such that p_ip_{i+1} is an edge for every $1 \leq i < k$. For every $1 \leq i < k$, the edge p_ip_{i+1} is an *edges of* P. Vertices p_1 and p_k are the *endpoints* of P, and $p_2 \dots p_{k-1}$ is the *interior* of P. P is referred to as a p_1p_k*-path*. For $1 \leq i \leq j \leq k$, we write $p_jPp_i := p_j \dots p_i$, $\mathring{P} := p_2 \dots p_{k-1}$, $\mathring{p}_jP\mathring{p}_i := p_{j+1} \dots p_{i-1}$.

A cycle C is a sequence of vertices $p_1p_2 \dots p_kp_1$, $k \geq 3$, such that $p_1 \dots p_k$ is a path and p_1p_k is an edge. Edges p_ip_{i+1}, for $1 \leq i < k$, and edge p_1p_k are called the *edges of* C. Let Q be a path or a cycle. The *length* of Q is the number of its edges. An edge $e = uv$ of G is a *chord* of Q if $u, v \in V(Q)$, but uv is not an edge of Q. A path or a cycle Q in a graph G is *chordless* if no edge of G is a chord of Q.

A graph G is a *complete k-partite* if $V(G)$ can be partitioned into k non-empty subsets $A_1, \dots, A_k$ such that, for $i = 1, \dots, k$, A_i is a stable set and, for any $\{i, j\} \subseteq \{1, \dots, k\}$, there is all possible edges between A_i and A_j. Sets A_i are called the *partitions' set* of G. G is noted $K_{a_1,\dots,a_k}$ where $a_i = |A_i|$ for $i = 1, \dots, k$. If $k = 2$ then G is said to be a *complete bipartite graph* and if $k = 3$, G is said to be a *complete tripartite graph*. The graph $K_{1,1,2}$ is called a *diamond*.

3 Preliminaries

We mentioned that $\mathcal{C}_1$ was already proved to be χ-bounded, we use this result for graphs in $\mathcal{C}_1$ that contain no K_4, which formally give:

Theorem 3.1 (Trotignon and Vušković [11]) *If $G \in \mathcal{C}_1$ and $\omega(G) \leq 3$, then $\chi(G) \leq 3$.*

Let us now explain a classical tool to prove χ-boundedness results for classes of graphs defined by forbidding induced structure and that is extensively use in this paper.

Let G be a graph. The *distance* between two vertices x, y of G is the length of a shortest xy-path. Let z be a vertex of G and let i be an integer. The *i-th level* of z is the set of vertices, denoted by $S_i(z, G)$, that are at distance exactly i from z in G. If no confusion is possible, we denote it by $S_i(z)$ in order to avoid too heavy notations. A *father* of a vertex $x \in S_i(z)$ is a vertex in $S_{i-1}(z)$ adjacent to x. For every pair of vertices x, y in $S_i(z)$, it is easy to see that there exists a chordless xy-path Q with internal vertices included in $z \cup S_1(z) \cup \dots \cup S_{i-1}(z)$ such that only the endpoints of $\mathring{Q}$ are in $S_{i-1}(z)$. Note that, as a consequence, the endpoints of $\mathring{Q}$ are the only vertices that have neighbors in $S_i(z)$. Such paths are called *unimodal paths* and are a key tool to find particular induced structures in a graph. In the remaining of the paper, the letter Q is reserved to denote unimodal paths and the following convention is followed: if x and y are two vertices in $S_i(z)$, Q_{xy} denotes a unimodal path with endvertices x and y.

The following general remark explains the reason why decomposing a graph into levels (as described above) is a very powerful tool to bound its chromatic number.

Remark 3.2 (Folklore) Let G be a graph and let z be a vertex of G. There exists an integer k such that $G[S_k(z)]$ has chromatic number at least $\lceil \chi(G)/2 \rceil$.

PROOF — For any $i \neq j$, there is no edges between a vertex of S_{2i} (resp. of S_{2i+1}) and of S_{2j} (resp. of S_{2j+1}). Indeed adjacent vertices are at distance one, so their level differ by at most one. So,

$$\chi(G) \leq max_{i\ even}\chi(G[S_i]) + max_{j\ odd}\chi(G[S_j]).$$

The result follows.

□

4 Graphs that do not contain a cycle with exactly two chords as induced subgraph

Let C be a cycle with exactly two chords $e_1 = a_1a_2$ and $e_2 = b_1b_2$. If e_1 and e_2 share an extremity, then e_1 and e_2 are said to be *V-chords* of C. If

a_1, a_2, b_1, b_2 are pairwise distinct and appear in the following order along C : a_i, b_j, a_k, b_l with $\{i,k\} = \{j,l\} = \{1,2\}$, then e_1 and e_2 are said to be *crossing chords* of C. A 2-cycle with V-chords (resp. crossing chords) is called a *V-cycle* (resp. an *X-cycle*) (see Figure 1). A 2-cycle that is not a V-cycle nor an X-cycle is a *parallel cycle*.

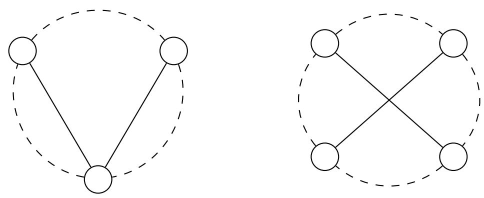

Figure 1: A V-cycle and an X-cycle.

The main result of this section is concerned with the class of (X-cycle, V-cycle)-free graphs that of course strictly contains the class of graphs with no 2-cycles. Indeed if a graph is (X-cycle, V-cycle)-free, it can contains a 2-cycle if the two chords of the cycle are parallel.

Theorem 4.1 *Every (X-cycle, V-cycle)-free graph G satisfies $\chi(G) \leq 6$.*

Note that we can bound the chromatic number by a constant here because K_4 is an X-cycle. The proof is built on two steps: Lemmas 4.2 and 4.3.

The first step consists in showing that we can decompose an (X-cycle, V-cycle)-free graph around a complete multipartite graph. More precisely we prove that if a graph is (X-cycle, V-cycle)-free then either it has a clique cutset, or it is a complete tripartite graph, or it is diamond-free. In the second part, we prove that (diamond, V-cycle, X-cycle)-free graphs have chromatic number at most 6 using Remark 3.2. The proof is somehow based on an induction on the number of chords. The induction is based on the result of Trotignon and Vušković about $\mathcal{C}_1$ (Lemma 3.1). We finally combine these two lemmas to prove Theorem 4.1.

Note that the first step actually gives us a decomposition theorem for (X-cycle, V-cycle)-free graphs, where the basic classes are complete multipartite graphs and (diamond, X-cycle, V-cycle)-free graphs. Anyway, to get a usable decomposition theorem, one should decompose (diamond, X-cycle, V-cycle)-free graphs that is a too complex class to be a basic class.

Lemma 4.2 *If G is an (X-cycle, V-cycle)-free graph, then either G has a clique cutset, or G is isomorphic to a complete tripartite graph, or G is diamond-free.*

PROOF — Assume by way of contradiction that G does not admit a clique cutset, G is not isomorphic to a complete tripartite graph and G contains a diamond. Since G has no clique cutsets, G is 2-connected. Let $H = K_{i,j,k}$ be a maximum (subject to its number of vertices) complete tripartite subgraph of G. Note $A = \{a_1, \dots, a_i\}$ (resp. $B = \{b_1, \dots, b_j\}$, resp. $C = \{c_1, \dots, c_k\}$) the set of the partition of H of cardinality i (resp. j, resp. k). Note that since G contains a diamond, one of the integers i, j, k is strictly greater than 1. Moreover, since G is K_4-free, H is an induced subgraph of G i.e. A, B and C are stable sets.

(1) A vertex $u \notin V(H)$ has at most one neighbor in H.

Assume by way of contradiction that some vertex $u \notin V(H)$ satisfies $d_H(u) \geq 2$. If u has a neighbor in A, B and C, say a_1, b_1 and c_1, then $ua_1b_1c_1$ is a K_4, a contradiction. So we may assume w.l.o.g. that u does not have any neighbor in C. Assume that u has a neighbor in A and a neighbor in B, say a_1 and b_1. By maximality of H, u has at least one non-neighbor in $A \cup B$. Assume w.l.o.g. that a_2 is a non-neighbor of u. Then $ua_1c_1a_2b_1u$ is a V-cycle with chords b_1a_1 and b_1c_1, a contradiction. So we may assume w.l.o.g. that u does not have any neighbor in B and thus have at least two neighbors in A, say a_1 and a_2. Then $ua_1b_1c_1a_2u$ is an X-cycle with chords a_1c_1 and a_2b_2, a contradiction. This proves (1).

Note that $G \neq H$ since otherwise G is a complete tripartite graph. Let K be a connected component of $G \setminus H$. By (1), vertices of K that have a neighbor in H, have a unique neighbor in H. Since G does not contain clique cutsets, $N_H(K)$ must contain two non-adjacent vertices. Therefore, K contains a chordless path $P = p_1 \dots p_k$ such that the neighbors of p_1 and p_k in H are two non-adjacent vertices. Among all such paths, let P be minimal. Assume w.l.o.g. that a_1 and a_2 are the neighbors of respectively p_1 and p_k in H. By minimality of P, no interior vertex of P has a neighbor in A.

If no interior vertex of P is adjacent to a vertex in B or C, then $a_1Pa_2b_1c_1a_1$ is an X-cycle with chords a_1b_1 and a_2c_1, a contradiction. Let i be the smallest integer such that p_i has a neighbor in B or C, say p_i is adjacent to b_1. Then no interior vertices of $p_1 \dots p_i$ is adjacent to any vertices of H and thus $a_1p_1Pp_ib_1a_2c_1a_1$ is a V-cycle with chords b_1a_1 and b_1c_1, a contradiction. □

Lemma 4.3 *If G is a (diamond, X-cycle, V-cycle)-free graph, then for any $z \in V(G)$ and for every integer k, $S_k(z) \in \mathcal{C}_1$.*

PROOF — Let G be a (diamond, X-cycle, V-cycle)-free graph and let $z \in V(G)$. Assume by way of contradiction that there exists an integer k such that $S_k(z)$ contains a 1-cycle C as an induced subgraph. Name a, b the extremities of the unique chord of C. The cycle C is edge-wise partitioned in two ab-path: P^l and P^r (for left and right path, see Figure 2).

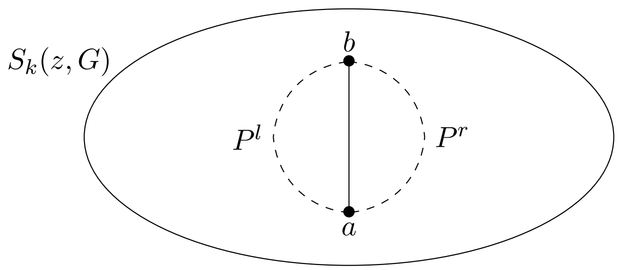

Figure 2: In $S_(z, G)$, the cycle C with a unique chord ab and the paths P^r and P^l

Observe that, since G is V-cycle-free, no vertex of G has four neighbors on an induced path. Moreover, no vertex x of G has four neighbors on an induced cycle. Indeed, either two consecutive neighbors are non adjacent, and then x has four neighbors on an induced path, or the cycle is a C_4 and then, G contains a diamond.

(1) For any $x \notin V(C)$, $d_C(x) \leq 3$ and, if x is adjacent to a or b, then $d_C(x) \leq 2$.

Let $x \notin V(C)$.

Suppose first that x is adjacent to a and that $d_C(x) \geq 3$. First assume that xb is an edge. Since $d_C(x) \geq 3$, x has another neighbor x_1 in C. We can assume w.l.o.g. that x_1 is on P^l. If x has no other neighbors on P^l, then $axbP^la$ is an X-cycle with chords xx_1 and ab. So x has another neighbor x_2 on P^l and then it has four neighbors in the chordless cycle aP^lba, a contradiction.

So xb is not an edge. Since $C \backslash \{b\}$ is an induced path, $d_C(x) = 3$. Denote by x_1 and x_2 the two other neighbors of x on C distinct from a. If x_1, x_2 are on P^l, then $aP^lx_1xx_2P^lbP^ra$ (resp. $aP^lx_1xx_2P^lba$) is a V-cycle (resp. an X-cycle) if x_1x_2 is not an edge (resp. is an edge) on a (resp. with chords ax and x_1x_2), a contradiction. Hence, by symmetry, we may assume that $x_1 \in P^l$ and $x_2 \in P^r$. Since G is diamond-free, a is not adjacent to both

x_1 and x_2. Assume w.l.o.g. x_1a is not an edge. Thus $x_1xaP^rbP^lx_1$ is an X-cycle with chords xx_2 and ab, a contradiction.

So the second outcome of the claim holds. Now, if x has at least 4 neighbors in C, then x is adjacent neither to a nor to b, and thus it has four neighbors on an chordless path, a contradiction. This proves (1).

(2) Vertices a and b do not have a common father.

Recall that, given two vertices x, y in $S_{k-1}(z)$, Q_{xy} denotes a unimodal path from x to y. And interior vertices of Q_{xy} are not adjacent to any vertex of C.

Assume by way of contradiction that there exists a vertex $x \in S_{k-1}(z)$ that is a common father to a and b. Let c be the neighbor of a on P^r and d be a father of c. By (1), $d \neq x$ and, since G is diamond-free, P^l and P^r have length at least 3 i.e. bc is not an edge.

If d is adjacent to a then $cdQ_{dx}xbac$ is a V-cycle on a. If d is adjacent to b then $cdQ_{dx}bac$ is an X-cycle with chords ax and bd. So d is adjacent neither to a nor to b.

Vertex d has at least one neighbor d_1 on $\mathring{P}^l$, otherwise $cdQ_{dx}bP^lac$ is a V-cycle on a. Moreover, d has a neighbor $d_2 \neq c$ on $\mathring{P}^r$ otherwise $cdQ_{dx}abP^rc$ is an X-cycle with chords ac and xb. By Claim 1, $d_C(x) \leq 3$, so $N_C(d) = \{c, d_1, d_2\}$.

If d_1b is not an edge, then $d_1dcP^rbaP^ld_1$ is an X-cycle with chords ac and dd_2. Otherwise $abxQ_{xd}d_1P^la$ is an X-cycle with chords bd_1 and ax, a contradiction. This proves (2).

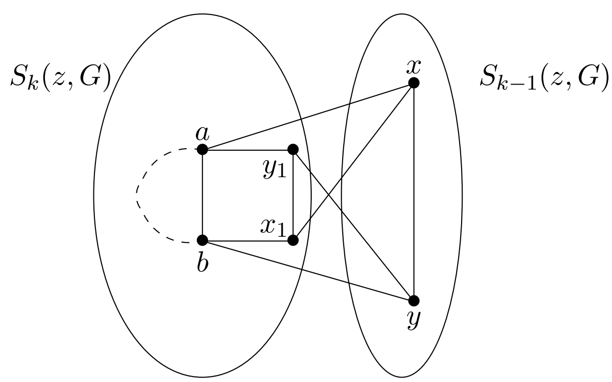

Figure 3: The particular graph of Claim 3, note that $P^r = ay_1x_1b$.

(3) Both a and b have a father of degree 1.

Let x and y be respectively a father of a and a father of b. By (2), $x \neq y$. Assume by way of contradiction that say x has a neighbor $x_1 \neq x$ in $\mathring{P}^r$. By (1), $N_C(x) = \{a, x_1\}$. If y has no neighbor in $\mathring{P}^r$, then $axQ_{xy}ybP^ra$ is an X-cycle with chords ab and xx_1, a contradiction. So y has a neighbor, say y_1, in $\mathring{P}^r$ and, by (1), $N_C(y) = \{b, y_1\}$.

Suppose first that $x_1 = y_1$. Since G is diamond-free, bx_1 and ax_1 cannot both be edges. So we may assume w.l.o.g. that bx_1 is not an edge. Then, $aP^rx_1xQ_{xy}yba$ is an X-cycle with chords ax and x_1y, a contradiction. So $x_1 \neq y_1$.

Now, if a, x_1, y_1 appear in this order along P^r, then $axQ_{xy}y_1P^rbP^la$ is a V-cycle on b, a contradiction. So a, y_1, x_1 appear in this order along P^r. If ay_1 is not an edge, then $axQ_{xy}yy_1P^rba$ is an X-cycle with chords xx_1 and by, a contradiction. So ay_1 is an edge and, symmetrically, bx_1 is an edge. If y_1x_1 is not an edge, then $ay_1yQ_{yx}x_1ba$ is an X-cycle with chords ax and by, a contradiction. So x_1y_1 is an edge (i.e. $ax_1y_1x_1b$ is a square). If xy is not an edge, then $abyy_1x_1xa$ is an X-cycle. So xy is an edge (see Figure 3).

Let c be the neighbor of a on P_l and let d a father of c. First assume that cb is an edge. Since G is diamond-free, d is adjacent neither to a nor to b. If x_1d is not an edge then $cdQ_{dx}xx_1bac$ is a X-cycle with chords cb and xa, a contradiction. Hence, by symmetry, d is adjacent to both x_1 and y_1 and then $axQ_{xd}dx_1y_1a$ is an X-cycle with chords dy_1 and xx_1.

Hence cb is not an edge. If d is adjacent to both x_1 and y_1, then $axQ_{xd}dx_1y_1a$ is an X-cycle with chords dy_1 and xx_1. If d is adjacent neither to x_1 nor to y_1, $cdQ_{dy}ybx_1y_1ac$ is a X-cycle with chords ab and yy_1. If d is adjacent to x_1 and not to y_1 then $cdQ_{dx}xx_1bac$ is a X-cycle with chords ax and dx_1. If d is adjacent to y_1 and not to x_1 then $cdQ_{dx}xx_1y_1ac$ is a X-cycle with chords yy_1 and xa This proves (3).

By (3), there exist two vertices x and y such that x is a father of a, y is a father of b and $d_C(x) = d_C(y) = 1$. Since G is diamond-free, P^r and P^l cannot be both of length two, so we may assume w.l.o.g. that P^l has length at least 3. Let c be the neighbor of a on P^l and d be a father of c. Note that $d \neq x$ and $d \neq y$. If $d_C(d) = 1$, then $axQ_{xd}dcP^lbP^ra$ is a V-cycle on a, a contradiction. Hence $d_C(d) \geq 2$.

Assume first d has a neighbor d_1 in P^r. If d_1 is the unique neighbor of d in P^r, then $cdQ_{dy}ybP^rac$ is an X-cycle with chords ab and dd_1 if $a \neq d_1$, or is a V-cycle on a if $d = a$ or $d = b$. So d has a second neighbor d_2 in P^r and thus, by (1), $N_C(d) = \{c, d_1, d_2\}$ and $\{d_1, d_2\} \subseteq \mathring{P}^r$. Assume w.l.o.g. that a, d_1 and d_2 appear in this order along P^r. Now, $axQ_{xd}dd_2P^rbP^la$ is an X-cycle with chords cd and ab, a contradiction. So d has no neighbors in

P^r and thus has some neighbors in $\mathring{P}^l$.

If d has exactly one neighbor d_1 in $\mathring{P}^l$, then $axQ_{xd}dcP^lba$ is an X-cycle with chords dd_1 and ac, a contradiction. So d has at least two neighbors in $\mathring{P}^l$ and, by (1), it has exactly two. Put $N_C(d) = \{c, d_1, d_2\}$. Now, $cdQ_{dy}ybP^lc$ is a V-cycle with chords dd_1 and dd_2, a contradiction. $\square$

We are now ready to prove Theorem 4.1, recall that this theorem states that *every (X-cycle, V-cycle)-free graph is 6-colorable.*

PROOF — Assume by way of contradiction that there is some (X-cycle, V-cycle)-free graphs that are not 6-colorable. Let G be minimal with this property. Suppose first that G contains a diamond. Since a complete tripartite graph is 3-colourable, by Lemma 4.2, G admits a clique cutset K. Let C_1 be a connected component of $G \setminus K$, and C_2 the union of all other components of $G \setminus K$. Put $G_1 = G[C_1 \cup K)]$ and $G_2 = G[C_2 \cup K)]$. If G_1 and G_2 are both 6-colourable, then G is 6-colourable, a contradiction. Therefore G_1 or G_2 is not 6-colourable, a contradiction to the minimality of G. So we may assume that G is diamond-free i.e., G is (diamond, X-cycle, V-cycle)-free.

Let z be a vertex of G. Since $\chi(G) = 7$, by Remark 3.2, there is an integer k such that $\chi(S_k(z)) \geq 4$. So, by Theorem 3.1, $S_k(z)$ contains a 1-cycle as an induced subgraph, a contradiction to Lemma 4.3. $\square$

5 Graphs that do not contain a cycle with exactly three chords as induced subgraph

The aim of this section is to prove that $\mathcal{C}_3$ is χ-bounded (Theorem 5.3).

The proof is divided into three parts, according to the clique number. First of all, we prove that every (triangle, 3-cycle)-free graph has chromatic number at most 24. Below, the constant 24 is denoted by c.

For graphs with clique number exactly 3, we prove that the chromatic number is at most $4c$. When the clique number is at least 4, then the chromatic number is close to the clique number. We prove that asymptotically the difference between them is at most one.

Let us state now the exacts statements of these 3 theorems. They are prove in Subsections 5.1, 5.2 and 5.3 respectively.

Theorem 5.1 *A (triangle, 3-cycle)-free graph has chromatic number at most c.*

Theorem 5.2 *A (K_4, 3-cycle)-free graph has chromatic number at most* $4c$.

Theorem 5.3 *A (3-cycle)-free graph has chromatic number at most* $\max(4c, \omega(G)+1)$.

Note that Theorem 5.3 says that, if a 3-cycle-free graph has a large enough clique (of size at least 96), then $\chi(G) \leq \omega(G)+1$. Moreover the Hajós join of two cliques shows this bound is tight. Let us recall what the Hajós join of two cliques is and prove it is 3-cycle-free.

Let us now describe the construction of the Hajós join of two K_k. Take two disjoint copies H_1 and H_2 of K_{k-1}, add a vertex x complete to H_1 and H_2 and two adjacent vertices a and b, such that a is complete to H_1 and b is complete to H_2. The obtained graph is the Hajós join of K_k and K_k and it is easy to check that it has clique number k and chromatic number $k+1$. Now, let us show it is 3-cycle-free. An ax-path with interior vertices in H_1 is either chordless or has at least two chords. Similarly, a bx-path with interior vertices in H_2 is either chordless or has at least two chords. So a cycle going through both H_1 and H_2 is either chordless, or has exactly two chords, or has more than four chords. Hence the graph is 3-cycle-free.

5.1 Clique number 2: proof of Theorem 5.1

Recall that Theorem 5.1 states that *a (triangle, 3-cycle)-free graph has chromatic number at most* $c = 24$.

To prove this result, we need the two following lemmas.

Lemma 5.4 *Let G be a (triangle, 3-cycle)-free graph. For every $z \in V(G)$ and every integer k, $S_k(z)$ is (V-cycle, triangle, 3-cycle)-free.*

Lemma 5.5 *Let G be a (V-cycle, triangle, 3-cycle)-free graph. For every $z \in V(G)$ and every integer k, $S_k(z)$ is (X-cycle, V-cycle, triangle, 3-cycle)-free.*

Before we prove these two lemmas, let us explain how they imply Theorem 5.1. Suppose there exists a (triangle, 3-cycle)-free graph G with $\chi(G) \geq 25$. Let z be a vertex of G. By Remark 3.2, there exists an integer k such that $\chi(G[S_k(z,G)]) \geq 13$. Put $H = G[S_k(z,G)]$. By Lemma 5.4, H is (V-cycle, triangle, 3-cycle)-free.

Let x be a vertex of H. By Remark 3.2, there exists an integer ℓ such $\chi(G[S_\ell(x,H)]) \geq 7$. So, by Theorem 4.1, $G[S_\ell(x,H)]$ contains an X-cycle as an induced subgraph (it cannot contain a V-cycle since it is an induced subgraph of H that is V-cycle-free) which contradicts 5.5.

5.1.1 Proof of Lemma 5.4

Recall that Lemma 5.4 states that, if G is a (triangle, 3-cycle)-free graph and z is a vertex of G, then for every integer k, $S_k(z,G)$ is (V-cycle, triangle, 3-cycle)-free.

PROOF — Let G be a (triangle, 3-cycle)-free graph and z a vertex of G. Assume by way of contradiction that there exists an integer k such that $S_k(z,G)$ contains an induced V-cycle C.

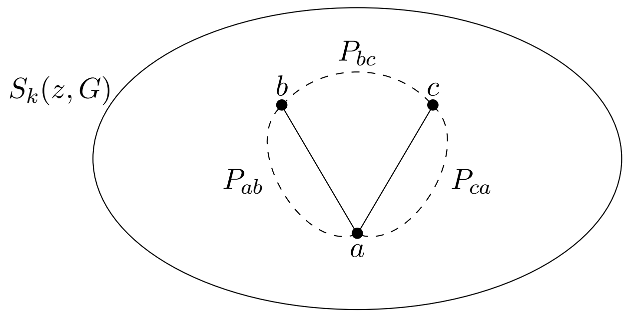

Figure 4: The V-cycle C in $S_k(z,G)$.

Let a be the unique vertex of C of degree 4 and let b, c be the two vertices of C of degree 3. Vertices a, b and c are called the *important* vertices of C. We denote by P_{ab} the path from a to b contained in $C\backslash\{ab, ac\}$ that avoids c. Paths P_{bc} and P_{ca} are defined similarly (see Figure 4). P_{ab}, P_{bc} and P_{ca} are called the *intervals* of C. By abuse of notation, P_{ab} will sometimes denote $V(P_{ab})$. Also, P_{ab} can be referred to as P_{ba} (and analogously for P_{bc} and P_{ca}). Moreover, if a vertex x is in $V(P_{ab})$, then the path $xP_{ab}b$ (resp. $xP_{ab}a$) can be referred to as P_{xb} or P_{bx} (resp. P_{xa} or P_{ax}). Given two vertices x and y in the same interval, the *external path from x to y* consists in the path from x to y in $C \setminus \{ab, ac\}$ passing through a, b and c. If x_1 and x_2 are two vertices of a path P that have a common father u, we say that x_1 and x_2 are *consecutive neighbors of u along P* if u has no neighbors in $\mathring{x}_1 P \mathring{x}_2$.

Note that adjacent vertices of C cannot have a common father otherwise G would contain a triangle. Also note that if a vertex has 5 neighbors on an induced path, there is a 3-cycle.

The proof consists in studying how a vertex not in C can attach on C and then using unimodal paths to get contradictions.

(1) If a vertex $u \notin V(C)$ satisfies $d_C(u) \geq 3$ and all but at most one neighbors of u are contained in an interval of C, then G contains a 3-cycle.

Let u be a vertex not in $V(C)$ such that: $d_C(u) \geq 3$ and all but at most one neighbor of u is contained in an interval of C. So, there exists two vertices u_1 and u_2 in $N_C(u)$ such that u_1 and u_2 are in the same interval of C and u has exactly one neighbor u_3 on the external path P from u_1 to u_2. Then $u_2uu_1Pu_2$ is a 3-cycle with chords ab, ac and uu_3 (note that, since G is triangle-free, u_1u_2 is not an edge). This proves (1).

(2) If $u \notin V(C)$ and $d_C(u) = 3$, then u has exactly one neighbor in each interval i.e. it has one neighbor in each of $\mathring{P}_{ab}$, $\mathring{P}_{bc}$ and $\mathring{P}_{ac}$.

This is immediate by (1). This proves (2).

(3) If $u \notin V(C)$, then u has at most 3 neighbors on $aP_{ab}bP_{bc}c$.

Since G is triangle-free, either a or b is not a neighbor of u. If u has at least 5 neighbors in $aP_{ab}bP_{bc}c$, then u has 5 neighbors on one of the chordless paths $\mathring{a}P_{ab}bP_{bc}c$ or $aP_{ab}bP_{bc}\mathring{c}$ which provides a 3-cycle, a contradiction.

So we may assume that u has exactly 4 neighbors in $aP_{ab}bP_{bc}c$ and w.l.o.g. that u has at least two neighbors in P_{ab}. Let u_1 and u_2 be two consecutive neighbors of u along P_{ab} such that a, u_1, u_2 appear in this order along P_{ab}. Then $u_1uu_2P_{u_2b}bP_{bc}caP_{au_1}u_1$ is a 3-cycle (recall that u_1u_2 is not an edge since G is triangle-free). This proves (3).

Note that by symmetry (3) also holds for $bP_{bc}cP_{ca}a$.

The next claim states the only way a vertex can have at least four neighbors in C.

(4) If $u \notin V(C)$ and $d_C(u) \geq 4$, then $d_C(u) = 4$ and $N_C(u) = \{b, c, y_1, y_2\}$ where y_1 is the neighbor of a in P_{ab} and y_2 is the neighbor of a in P_{ca}.

Let $u \notin V(C)$ and suppose $d_C(u) \geq 4$.

If u has at least three neighbors in P_{bc}, then it has at least four neighbors either in $P_{ab}P_{bc}$ or in $P_{bc}P_{ac}$, which contradicts (4). So u has at most two neighbor in P_{bc}.

Case 1 : u has exactly two neighbors, u_1, u_2 say, on P_{bc}.
Assume w.l.o.g. that b, u_1, u_2, c appear in this order along P_{bc}. By (3), u has at most one neighbor on $aP_{ab}\mathring{b}$ and at most one neighbor on $\mathring{c}P_{ca}a$. Since $d_C(u) \geq 4$, both neighbors exists. Moreover both are distinct from a otherwise there would be 4 neighbors on $P_{ab}P_{bc}$ or $P_{bc}P_{ca}$, contradicting (3). Denote by y_1 (resp. y_2) the neighbor of u in $P_{ab}\mathring{b}$ (resp. in $\mathring{c}P_{ca}$).

If $u_1 \neq b$ then $y_1uu_1P_{u_1c}cP_{ca}P_{ay_1}y_1$ has 3 chords, namely uy_2, uu_2 and ac. So $u_1 = b$ and, by symmetry, $u_2 = c$. If y_2a is not an edge then

$y_2uu_1P_{u_1a}acP_{cy_2}y_2$ is a 3-cycle with chords ab, uy_1 and uc. So ay_2, and by symmetry ay_1, are edges, so the outcome holds.

Case 2 : u has exactly one neighbor, u_3 say on P_{bc}.
Since $d_C(u) \geq 4$, u has at least 3 neighbors on $\mathring{b}P_{ba}P_{ac}\mathring{c}$. W.l.o.g. u has at least two neighbors in $P_{ab}\mathring{b}$. By (3), u has exactly two neighbors, u_1, u_2 say, in $P_{ab}\mathring{b}$. Assume that a, u_1, u_2 appear in this order along $P_{ab}\mathring{b}$. Let u_4 be the neighbor of u that is closest from a in $\mathring{P}_{ca}$ (u_4 exists since $d_C(u) \geq 4$). If $u_3 \neq c$ then $u_4uu_3P_{u_3b}bP_{ba}P_{ac}u_4$ is a 3-cycle with chords ab, uu_1 and uu_2. So $u_3 = c$.

Note that a is not a neighbor of u otherwise a, c, u would be a triangle. So $N_C(u) = \{u_1, u_2, c, u_4\}$, otherwise u would have 5 neighbors on the chordless path $V(C) \setminus \{a\}$, i.e. there would be 3-cycle. Hence $u_2ucP_{ca}P_{au_2}u_2$ is a 3-cycle with chords uu_1, ac and uu_4, a contradiction.

Case 3 : u no neighbor on P_{bc}.
Since $d_C(u) \geq 4$, we may assume w.l.o.g. that u has at least two neighbors, u_1, u_2 say, on $P_{ab}\mathring{b}$. By (1), u has at least two other neighbors u_3, u_4 on $\mathring{c}P_{ca}$. Moreover u has no other neighbors in C since otherwise u would have 5 neighbors on the chordless path $V(C) \backslash P_{bc}$. By (1), u_1, u_2, u_3, u_4 are distinct from a. Assume w.l.o.g. that u_2, u_1, a, u_3, u_4 appear in this order along $\mathring{b}P_{ba}P_{ac}\mathring{c}$.

If au_3 is an edge then $u_3uu_2P_{u_2a}acP_{ca}$ is a 3-cycle with chords au_3, uu_4 and uu_1. So we may assume au_3 is not an edge. If u_2b is an edge then $u_2uu_4P_{u_4c}cP_{cb}baP_{au}u_2$ is a 3-ycle with chords uu_1, ac and u_2b. So u_2b is not an edge and hence, $u_2uu_3P_{u_3c}cP_{cb}baP_{au_2}u_2$ is a 3-cycle with chords uu_1, uu_4 and ac, a contradiction.

This proves (4).

(5) Let y be a father of a vertex of C. Then $d_C(y) \leq 3$.

Let y_1 be the neighbor of a in P_{ab} and y_2 be the neighbor of a in P_{ca}. Let y be the father of a vertex in C. Suppose for contradiction that $d_C(y) \geq 4$ By (4), $N_C(y) = \{b, c, y_1, y_2\}$.

Let e be the neighbor of b on P_{ab}. Note that $e \neq y_1$ since otherwise aby_1 is a triangle. Let f be a father of e. By (4), $d_C(f) \leq 3$. If f has no neighbor in $P_{ac} \cup P_{ab} \setminus \{e, b\}$, then $efQ_{fy}ycP_{ca}P_{ab}e$ is a 3-cycle with chords yy_1, yy_2 and ac. So f has at least one neighbor in $P_{ac} \cup P_{ab} \setminus \{e, b\}$.

Assume that f has at least one neighbor f_1 in $P_{ab} \setminus \{e, b\}$. Then it is the only one, otherwise $d_C(f) = 3$ and all neighbors of f are in the same interval contradicting (2). Then $efQ_{fy}baP_{ae}e$ is a 3-cycle with chords eb, yy_1 and

ff_1. So f has no neighbors in $P_{ab} \setminus \{e, b\}$.

Hence f has at least one neighbor f_1 in $P_{ca} \setminus \{a\}$ and it is the only one by (2). If f has no neighbor on $\mathring{P}_{bc}$ then $efQ_{fy}yy_2P_{ac}P_{ce}e$ has chords yb, yc and ff_1. So, f has at least one neighbor f_2 in $\mathring{P}_{bc}$, and it is the only one by (2).

If fy is an edge then $f_2feP_{ey_1}y_1yy_2P_{y_2c}P_{cf_2}$ is a 3-cycle with chords fy, ff_1 and yc. Otherwise $eff_2P_{f_2b}byy_2aP_{ae}$ is a 3-cycle with chords, eb, ab and yy_1. This proves (5).

(6) If x is a father of an important vertex of C, then $d_C(x) \leq 2$.

Let x be a father of an important vertex of C. By (5), $d_C(x) \leq 3$. By (2), the father of an important vertex cannot have exactly three neighbors in C. So $d_C(x) \leq 2$. This proves (6).

(7) Let e be the neighbor of a on P_{ab} and let f be a father of e. Then $d_C(f) \leq 2$.

Assume for contradiction that $d_C(f) = 3$. By (5), f has a exactly one neighbor, f_1 say, in $\mathring{P}_{bc}$ and exactly one, f_2 say, in $\mathring{P}_{ca}$. All of them are distinct from important vertices since fathers of important vertices have at most two neighbors on C.

Let x be a father of a. If x has no neighbor on $\mathring{a}P_{ab}P_{bc}$, then $efQ_{fx}acP_{cb}P_{be}e$ is a 3-cycle with chords ab, ae and ff_1. So x has a neighbor, x_1 say in $\mathring{a}P_{ab}P_{bc}$ and, by (6), $N_C(x) = \{a, x_1\}$. If $x_1 \in \mathring{P}_{ab}P_{bf_1}$, then $axQ_{xf}f_1P_{f_1b}P_{ba}a$ is a 3-cycle with chords ab, ef and xx_1. So $x_1 \in f_1P_{f_1c}c$ and then $axQ_{xf}ff_1P_{f_1c}P_{ca}a$ is a 3-cycle with chords ac, ff_2 and xx_1 a contradiction. This proves (7).

(8) Let x be a father of a. Then $d_C(x) = 2$.

By (6), $d_C(x) \leq 2$. So we may assume by way of contradiction that $d_C(x) = 1$. Let e be the neighbor of a on P_{ab} and let f be a father of e. Finally let y be a father of b.

If $d_C(f) = 1$, then $axP_{xf}eP_{eb}P_{bc}P_{ca}a$ is a 3-cycle with chords ae, ab, ac. So $d_C(f) \geq 2$ and thus, by (7), $d_C(f) = 2$. If the second neighbor f_1 of f is on $P_{ab}P_{bc}$, then $axQ_{xf}eP_{eb}P_{bc}ca$ is a 3-cycle with chords ae, ab and ff_1. So $f_1 \in \mathring{P}_{ca}$.

Note that eb is not an edge since otherwise aeb is a triangle. If y has a neighbor in $\mathring{b}P_{bc}P_{ca}e$, then $byQ_{yf}feaP_{ac}P_{cb}b$ is a 3-cycle with chords ab, ac and ff_1. So y has a neighbor, say y_1, in $\mathring{b}P_{bc}P_{ca}e$ and by (6), $N_C(y) = \{b, y_1\}$. If $y_1 \neq e$, then $axQ_{xy}bP_{bc}P_{ca}$ is a 3-cycle with chords ab, ac and yy_1. Hence

$y_1 = e$ which contradicts (7). This proves (8).

We now have proved enough claims to finish the proof. Let x be a father of a. By (8), $d_C(x) = 2$. Let x_1 be the neighbor of x distinct from a on C. Let e be the neighbor of a on P_{ab} and let f be a father of e. Finally let y be a father of b. By symmetry we may assume that $x_1 \in P_{bc}P_{ca}\mathring{a}$.

If $d_C(y) = 1$, then $axQ_{xy}bP_{bc}P_{ca}$ would be 3-cycle with chords ab, ac and xx_1. So $d_C(y) \geq 2$ and by (6), $d_C(y) = 2$. Let y_1 be the neighbor of y distinct from b on C.

Assume that both x_1, y_1 are on P_{ca}. If a, y_1, x_1 appears in this order along P_{ac} then $x_1P_{x_1a}P_{ab}yQ_{yx}x_1$ is a 3-cycle with chords ax, ab and yy_1 ($x_1 \neq c$ since otherwise acx is a triangle). So a, x_1, y_1 appear in this order along P_{ac} and $x_1 \neq y_1$. In particular y_1a is not an edge and so $axQ_{xy}yy_1P_{y_1c}P_{cb}P_{ba}a$ is a 3-cycle with chords ab, ac and yb. Moreover, if both x_1, y_1 are on P_{bc} then $axQ_{xy}bP_{bc}a$ is a 3-cycle with chords ab, xx_1 and yy_1. So, either $x_1 \in \mathring{P}_{bc}$ and $y_1 \in \mathring{P}_{ca}$ or $x_1 \in \mathring{P}_{ca}$ and $y_1 \in \mathring{P}_{bc}$.

If $x_1 \in \mathring{P}_{bc}$ and $y_1 \in \mathring{P}_{ca}$ and then $x_1xQ_{xy}y_1P_{y_1a}P_{ab}P_{bx_1}$ is a 3-cycle with chords ax, by and ab. Thus $x_1 \in \mathring{P}_{ca}$ and $y_1 \in \mathring{P}_{bc}$ and then $x_1xQ_{xy}y_1P_{y_1b}P_{ba}P_{bx_1}$ is a 3-cycle with chords ax, by and ab, a contradiction that put an end to the proof. $\square$

5.1.2 Proof of Lemma 5.5

Recall that Lemma 5.5 states that, if G is a (triangle, 3-cycle, V-cycle)-free graph and z is a vertex of G, then for every integer k, $S_k(z, G)$ is X-cycle-free.

PROOF — Let G be a (triangle, 3-cycle, V-cycle)-free graph, z a vertex of G and suppose for contradiction that there exists an integer k such that $S_k(z, G)$ contain an X-cycle C as an induced subgraph.

Let ac and bd be the two chords of C and assume that a, b, c, d, appear in this order along C. Vertices a, b, c, and d are called *important vertices* of C. Two important vertices that do not form a chord of C are said to be *consecutive*. An *interval* is an induced path on $C \setminus \{ac, bd\}$ between two consecutive important vertices. They are denoted by P_{ab}, P_{bc}, P_{cd} and P_{da} (see Fig. 5.1.2). Note that two intervals share at most one vertex. By abuse of notation, P_{ab} will sometimes denotes $V(P_{ab})$. Also, P_{ab} can ve referred to as P_{ba} (and analogously for P_{cb}, P_{dc} and P_{da}). Given two vertices x and y in the same interval, the *external path from x to y* consists in the path from x to y in $C \setminus \{ac, bd\}$ passing through a, b, c and d.

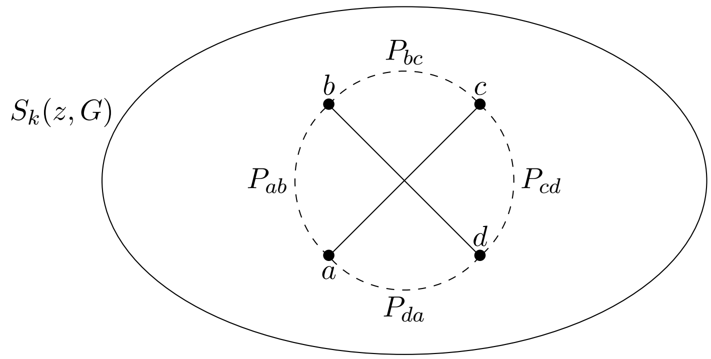

Figure 5: The X-cycle C in $S_k(z, G)$.

The proof is divided in two parts. First we prove that no neighbor of the graph has degree larger than 3 on C. We then study more specifically fathers of important vertices and prove that they are neither of degree 3, nor 2 nor 1.

(1) If a vertex $u \notin V(C)$ satisfies $d_C(u) \geq 3$ and all but at most one neighbors of u are contained in an interval of C, then G contains a 3-cycle.

Let u be a vertex not in $V(C)$ such that: $d_C(u) \geq 3$ and all but at most one neighbors of u are contained in an interval of C. So, there exists two vertices u_1 and u_2 in $N_C(u)$ such that u_1 and u_2 are in the same interval of C and u has exactly one neighbor u_3 on the external path P from u_1 to u_2. Then $u_2uu_1Pu_2$ is a 3-cycle with chords ab, ac and uu_3 (note that, since G is triangle-free, u_1u_2 is not an edge). This proves (1).

(2) Every vertex of $u \notin V(C)$, satisfies $d_C(u) \leq 3$. Moreover, if $d_C(u) = 3$, then no interval contains at least two neighbors of u.

Let us first prove a fact. If a vertex v has (at least) 4 neighbors on a path P that has at most one chord, then G contains a V-cycle or a 3-cycle. Indeed let v_1, v_2, v_3, v_4 be consecutive neighbors of v on P. Then $C' = vv_1Pv_4v$ has chords vv_2 and vv_3. If v_1Pv_4 is chordless then C' is a V-cycle otherwise C' is a 3-cycle.

Assume that a vertex $u \notin C$ satisfies $d_C(u) \geq 4$. Since G is triangle free, u is not adjacent to both a and c. By symmetry we can assume that u is not adjacent to a, then u has 4 neighbors on the path $\mathring{a}P_{ab}P_{bc}P_{cd}P_{da}\mathring{a}$ with at most one chord, contradicting the fact proved above.

If $d_C(u) = 3$, then (1) ensures that the neighbors of u are in distinct intervals. This proves (2).

So any vertex $x \notin V(C)$ satisfying $d_C(x) = 3$ is adjacent to at most one important vertex. Indeed two important vertices either are opposite (which would create a triangle), or are in a same interval (which would contradict (2)).

(3) Two adjacent vertices of C do not both have a father of degree one on C.

Let u and v be two adjacent vertices of C. Suppose for contradiction that u (resp. v) admits a father u' (resp. v') such that $d_C(u') = 1$ (resp. $d_C(v') = 1$). We denote by P the external path from v to u. Then $uu'Q_{u'v'}vuP$ is a 3-cycle with chords, uv, ac and bd. This proves (3).

(4) Let x be a father of an important vertex. Then $d_C(x) \leq 2$.

Assume by contradiction that a father x of a satisfies $d_C(x) = 3$. By (2), x has exactly one neighbor, x_1 say, on $\mathring{P}_{bc}$, and exactly one neighbor, x_2 say, on $\mathring{P}_{cd}$. Indeed a is in both P_{ab} and P_{da} and (2) ensures that there is no two neighbors of x in the same interval. Note that it implies that neither bc nor cd are edges.

First assume that ab is an edge. Let y be a father of c. If $d_C(y) = 1$, then $axQ_{xy}cP_{cb}dP_{da}$ is a 3-cycle with chords ab, ac and xx_1. So $d_C(y) \geq 2$. If $d_C(y) \geq 3$, then by (2), $d_C(y) = 3$ and y must have a vertex in $\mathring{P}_{ab}$ which is impossible since we assumed that ab is an edge. So $d_C(y) = 2$. If y_1 is on $P_{ab}P_{bc}$, then $axQ_{xy}cP_{cb}P_{ba}a$ is a 3-cycle with chords ac, xx_1 and yy_1, and if y_1 is on $P_{ad}P_{dc}$, then $axQ_{xy}cP_{cd}P_{da}a$ is a 3-cycle with chords ac, xx_2 and yy_1, a contradiction. So in the following we assume that ab, and by symmetry ad, are not edges.

If bx_1 is an edge then $axx_1P_{x_1c}P_{cd}bP_{ba}a$ is a 3-cycle with chords bx_1, xx_2 and ac. So bx_1, and by symmetry dx_2, are not edges.

Let e be the neighbor of a on P_{ab} and f be a father of e. The cycle $C' = efQ_{fx}x_2P_{x_2c}aP_{ad}bP_{be}e$ has chords xa and xe so it must admit other chords otherwise it is a V-cycle. We already showed that dx_2 nor bc are edges and that the only neighbors of x in C are a, x_1 and x_2. So others chords are due to neighbors of f. Moreover, f must have at least two neighbors that create chords in C', otherwise C' would be a 3-cycle. So, by (2), f has one neighbor on $P_{cd}\mathring{d}$ and one neighbor on $\mathring{d}P_{da}$. Let f_1 be the neighbor of f in $P_{cd}\mathring{d}$. Then $axQ_{xf}eP_{eb}dP_{dc}a$ is a 3-cycle with chords ae, xx_2 and ff_1 (note that ad is not an edge since f has a neighbor on $\mathring{P}_a$). This proves (4).

(5) If ab is an edge, fathers of a have degree exactly two.

Assume by contradiction that a father x of a satisfies $d_C(x) \neq 2$. So, by (4), $d_C(x) = 1$. Let z be a father of c. The cycle $C' = axQ_{xz}cP_{cb}dP_{da}$ has chords ac, ab which, with no additional chords, provides a V-cycle. By (4), $d_C(z) \leq 2$, so C' has at most one other chord due to neighbors of z. Moreover, if cd is an edge, it is a chord of C'. Since C' cannot be a V-cycle nor a 3-cycle, cd is an edge and z has another neighbor z_1 on C'. Since both ab and cd are edges, $z_1 \in P_{bc}$ or in $z_1 \in P_{da}$.

Assume first that $z_1 \in P_{bc}$. If $z_1 \neq b$, then $axQ_{xz}z_1P_{z_1c}dP_{da}$ is a V-cycle with chords zc and ac. So $z_1 = b$. Let e be the neighbor of a in P_{ad} (note that $e \neq d$ otherwise abd is a triangle) and let f be a father of e. Since $d_C(x) = 1$, $d_C(f) \geq 2$ by (3). Such a neighbor, called f_1, is unique, otherwise, since ab and cd are edges, at least two neighbors of f would be in the same interval, contradicting (2). Note that $f_1 \neq a$ otherwise aef_1 is a triangle. Then $efQ_{fz}bP_{bc}dP_{de}$ is a 3-cycle with chords ff_1, zc and bd. So $z_1 \notin P_{bc}$.

Thus, $z_1 \in P_{ad}$. Note that $z_1 \neq a$ since otherwise acz is a triangle. If az_1 is not an edge then $axQ_{xz}z_1P_{z_1d}cP_{cb}a$ is a 3-cycle with chords ac, bd and cz. So az_1 is an edge. Let y be a father of b. We have $d_C(y) \leq 2$ by (4). Then $d_C(y) = 2$ by (3) since $d_C(x) = 1$ and ab is an edge. Moreover, ay is not an edge otherwise aby is a triangle. So y has a neighbor y_1 in $\mathring{b}P_{bc}dP_{dz_1}$. Therefore $byQ_{yz}z_1P_{z_1d}cP_{cb}$ is a 3-cycle with chords zc, bd and yy_1. This proves (5).

(6) If an important vertex has a father of degree one on C, then every father of every important vertex has degree one on C.

Assume w.l.o.g. that a father x of a satisfies $d_C(x) = 1$. We show that it implies that every father of b are of degree one in C which, by symmetry, prove the claim.

Let y be father of b and assume for contradiction that $d_C(y) \neq 1$. Note that by (5), neither ab nor ad are edges. By (4), $d_C(y) = 2$. Let y_1 be the neighbor of y on C distinct from b. Let b_1 be the element of $\{y_1, b\}$ that is nearest from a in P_{ab} and which is distinct from a and let b_2 the other one. Such a vertex exists since b satisfies the conditions. If ab_1 is not an edge, then $axQ_{xy}b_1P_{b_1b}P_{bc}P_{cd}P_{da}$ has 3 chords ac, bd and yb_2. So we may assume that ab_1 is an edge, since ab is not an edge, $b_1 = y_1$.

Let z be a father of d. The cycle $C' = dzQ_{zy}y_1P_{y_1b}P_{bc}P_{cd}$ is, with no additionnal chord, a V-cycle with chords yb, bd. Since $d_C(z) \leq 2$ by (4), there is at most one chord with extremity z, which provides a 3-cycle. This proves (6).

(7) Fathers of important vertices have degree exactly two on C.

Let us prove it by contradiction. By (6), we can assume that all fathers of all important vertices have degree one on C. And by (5), none of ab, bc, cd and da are edges. Let u be a neighbor of a in P_{ab}. Let x be a father of a and y be a father of u. By (3), $d_C(y) \neq 1$ since $d_C(x) = 1$.

By (2), $d_C(y) \leq 3$. If $d_C(y) = 3$ then, by (2) and (6), the neighbors of y are in the interior of distinct intervals. Assume that y has a neighbor in $\mathring{P}_{cd}$ and in $\mathring{P}_{da}$. Let y_1 be the neighbor of y on $\mathring{P}_{cd}$. By (6), a father y' of b satisfies $d_C(y') = 1$. Hence $by'Q_{y'y}y_1P_{y_1d}P_{da}P_{ab}$ has 3 chords: bd and two chords with extremity y. It is easy to see that, since fathers of every important vertex are of degree one in C, we get a contradiction when $d_C(y) = 3$. So $d_C(y) = 2$.

Let u' be the other neighbor of a and let z be a father of u'. Assume first that u' has a father z distinct from y. By symmetry with y, $d_C(z) = 2$. So $u'zQ_{zy}uP_{ub}P_{bc}P_{cd}P_{du'}u'$ has 3 chords: two chords are given by the other neighbors of y and z and the third one is bd. So y is adjacent to u'.

Let w be the neighbor of c in P_{dc} and w' the neighbor of c on P_{cb}. For symmetric reason why y is a father of both u and u', there exists a vertex f that is the father of both w and w'. and that is of degree 2 in C. Then $wfQ_{fy}u'aP_{ab}P_{bc}w$ is a 3-cycle with chords fw', yu and ac (recall that both v and w are distinct from d since none of ad, cd are edges). This proves (7).

We are now armed to finish the proof! By (7), we may assume that fathers of every important vertex have exactly degree 2 on C. Let x and y be some fathers of a and c respectively. Since G is triangle-free, $x \neq y$. Let us denote by x_1 and y_1 the other neighbors of respectively x and y. If x_1 and y_1 are on $P_{ab}P_{bc}$, then $axQ_{xy}cP_{cb}P_{ba}$ is a 3-cycle with chords ac, xx_1 and yy_1, a contradiction. So x_1 and y_1 cannot both be on $P_{ab}P_{bc}$ and, symmetrically, they cannot be on $P_{cd}P_{da}$.

So, we may assume w.l.o.g. that x_1 is on $P_{ab}P_{bc}$ and that y_1 is on $P_{cd}P_{da}$. If $x_1 \in \mathring{P}_{ab}$ then $x_1xQ_{xy}cP_{cd}P_{da}P_{ax_1}$ is a 3-cycle with chords ac, ax and yy_1. Thus x_1 is on P_{bc} and by symmetry y_1 is on P_{da}. More generally, we showed that no father of an important vertex has its second neighbor on the interior of and interval adjacent containing it. If ab and cd are both not edges, then $axP_{xy}cP_{cb}dP_{da}$ is a 3-cycle with chords ac, xx_1, yy_1. So either ab is an edge, or cd is an edge, or both are edges.

Assume w.l.o.g. that ab is an edge. A father w of d has it second neighbor w_1 neither in P_{cd} nor in P_{da} since no father of an important vertex has its second neighbor on the interior of intervals adjacent to it. Since ab is an edge, $w_1 \in P_{bc}$. Now, a father z of b has a unique second neighbor z_1 on $\mathring{P}_{ad}$

(by applying the first part of the proof on b, d instead of a, c). If a, y_1, z_1 appears in this order along P_{ad} then $bzQ_{zy}y_1P_{y_1d}P_{dc}P_{cb}$ is a 3-cycle with chords bd, zz_1 and yc. So a, z_1, y_1 appear in this order along P_{ad} and, in particular, z_1d is not an edge. Symmetrically, b, x_1, w_1 appear in this order along P_{cb} and cx_1 is not an edge. Finnaly $x_1xQ_{xz}z_1P_{z_1a}cP_{cd}bP_{bx_1}$ is a 3-cycle with chords ab, xa and zc, a contradiction. $\square$

5.2 Clique number 3 : proof of Theorem 5.2

Recall that Theorem 5.2 ensures that, if G is a (K_4, 3-cycle)-free graph then $\chi(H) \leq 4c$.

The proof of Theorem 5.2 is organized as follows. First of all, we prove (see Lemma 5.6) that any (K_4, 3-cycle)-free graph with chromatic number at least $2c$ contains either an butterfly as an induced subgraph or a dragonfly as an induced subgraph (see Figure 6). Note that the proof is based on Theorem 5.1.

We then prove that if a graph G is (K_4, 3-cycle)-free and x is a vertex of G, then for any integer k, $S_k(z, G)$ is (dragonfly,butterfly)-free (see Lemmas 5.7 and 5.8).

At the very end, we combine these two result to get the proof of Lemma 5.2.

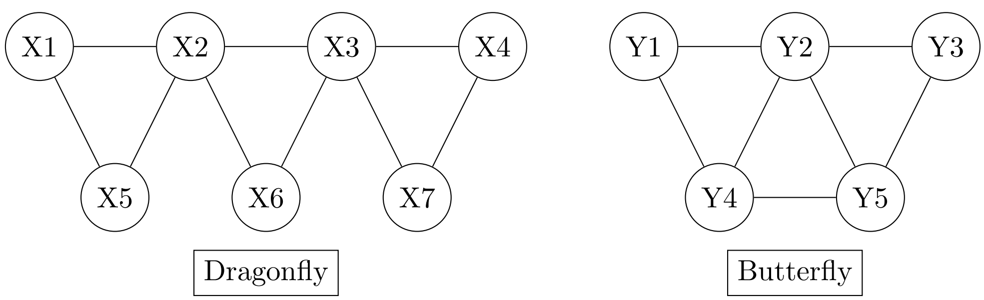

Figure 6: The dragonfly and the butterfly

Lemma 5.6 *Let G be a (K_4, 3-cycle)-free graph with $\chi(G) > 2c$. Then G contains a dragonfly or a butterfly as an induced subgraph.*

PROOF — All along the proof, the notations of the vertices of dragonfly and butterfly will fit with notations of Figure 6. We first prove that G admits a

dragonfly or a butterfly as a subgraph. We then prove that it is induced.

(1) G admits a dragonfly as a subgraph.

Let $T \subseteq V(G)$ be a minimal (by inclusion) subset of vertices such that $G \setminus T$ is triangle-free. By Theorem 5.1, $G \setminus T$ is c-colorable. If $G[T]$ is triangle-free, then $G[T]$ is c-colorable and thus G is $2c$-colorable, a contradiction. Thus, we may assume that $G[T]$ admits a triangle $x_2x_3x_6$. By minimality of T, $(G \setminus T) \cup \{x_2\}$ admits a triangle containing x_2, say $x_1x_2x_5$. Similarly, $(G \setminus T) \cup \{x_3\}$ contains a triangle containing b, say $x_3x_4x_7$.

If $\{x_1, x_5\} = \{x_4, x_7\}$, then $x_1x_2x_3x_5 = K_4$, a contradiction. So $\{x_1, x_5\} \neq \{x_4, x_7\}$.

Assume now that $|\{x_1, x_5\} \cap \{x_4, x_7\}| = 1$ and, w.l.o.g., assume that $x_1 = x_7$ (see Figure 5.2). The cycle $C = x_1x_5x_2x_6x_3x_4x_1$ is, if no additional chords exist, a 3-cycle with chords x_1x_2, x_1x_3 and x_2x_3. So C must have at least one more chord. Since G is K_4-free, x_1x_6, x_2x_4 and x_3x_5 are not edges. There remains only 3 possible chords, namely x_4x_5, x_5x_6 and x_4x_6. If say $x_4x_5 \in E(G)$, then $x_1x_2x_5x_4x_3x_1$ is a 3-cycle, a contradiction. So x_4x_5 is not an edge and, by symmetry, x_5x_6 and x_4x_6 are not edges.

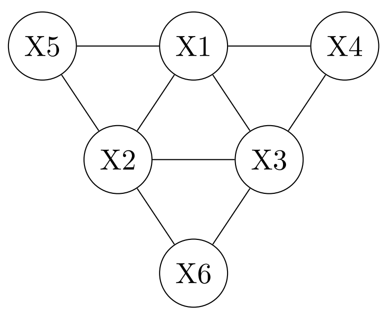

Figure 7: Figure in the proof of Claim (1).

So, $\{x_1, x_5\} \cap \{x_4, x_7\} = \emptyset$ and thus $G[\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}]$ contains a dragonfly as a subgraph. This proves (1).

(2) G contains either a dragonfly or a butterfly as an induced subgraph.

Observe first that, if G contains a butterfly (see Figure 6 for the name of its vertices), then it is induced. Indeed, since G is K_4-free, if it is not induced then $y_1y_3 \in E(G)$ and then $y_1y_4y_5y_3y_2y_1$ is a 3-cycle, a contradiction.

By (1), G admits a dragonfly as a subgraph, name it H and refer to Figure 6 for the name of its vertices. We may assume that H is not induced, otherwise we are done. If there exists an edge with one extremity in $\{x_1, x_5\}$

and the other one in $\{x_3, x_6\}$, then $G[\{x_1, x_2, x_3, x_5, x_6\}]$ contains a butterfly as a subgraph and thus as an induced subgraph. So there is no edges with one extremity in $\{x_1, x_5\}$ and the other one in $\{x_3, x_6\}$ and, by symmetry, there is no edges with extremities in $\{x_4, x_7\}$ and the other one in $\{x_2, x_6\}$.

So there exists some edges with one extremity in $\{x_1, x_5\}$ and one extremity in $\{x_4, x_7\}$. By symmetry, we may assume w.l.o.g. that $x_5x_7 \in E(G)$. If it is the only one then, $x_1x_2x_6x_3x_4x_7x_5x_1$ is a 3-cycle, a contradiction. So it is not the only one and thus, some of x_1x_4, x_1x_7 or x_4x_5 are edges of G. If there is exactly one more, then in the three cases $x_1x_2x_3x_4x_7x_5x_1$ is a 3-cycle, a contradiction. So, there is at least two more and there is actually exactly two more, otherwise $x_1x_4x_5x_7 = K_4$. By symmetry between x_1x_7 and x_4x_5, we may assume w.l.o.g. that $x_4x_5 \in E(G)$. So one of the edges x_1x_4 or x_1x_7 exists, but in both cases the cycle $x_2x_6x_3x_4x_7x_5x_2$ is a 3-cycle, a contradiction. This proves (2). $\square$

Lemma 5.7 *Let G be a (K_4, 3-cycle)-free graph and let z be a vertex of H. Then, for every integer i, $G[S_i(z)]$ is dragonfly-free.*

PROOF — Assume by way of contradiction that there exists an integer i such that $G[S_i(z)]$ contains a dragonfly as an induced subgraph. Name it H and refer to Figure 6 for the name of its vertices. Let u be a father of x_5 and v be a father of x_7.

The two next claims examine what are the possible neighborhoods of u and v in H.

(1) $N_H(u) \in \{\{x_5\}, \{x_5, x_1\}, \{x_5, x_2\}, \{x_5, x_3\}, \{x_5, x_6\}, \{x_1, x_3, x_5, x_6\}\}$.

First note that u cannot have exactly two neighbors in $\{x_1, x_2, x_3, x_6\}$. Indeed, u cannot see both x_1 and x_2 since otherwise there is a K_4. Thus w.l.o.g. x_6 is a neighbor of u and then $ux_5x_1x_2x_3x_6u$ is a 3-cycle.

Assume now that ux_2 is an edge. Since G is K_4-free, ux_1 is not an edge. Both ux_3, ux_6 are not edges since G is K_4-free. So none of ux_3, ux_6 is an edge since otherwise u has exactly two neighbors in $\{x_1, x_2, x_3, x_6\}$. If ux_7 is an edge then $ux_5x_1x_2x_6x_3x_7u$ is a 3-cycle. So ux_7 and by symmetry ux_4 are not edges. So if ux_2 is an edge, then $N_H(u) = \{x_5, x_2\}$ and one of the outcome holds. So, we may assume from now on that ux_2 is not an edge.

Assume that ux_7 is an edge. By symmetry between x_5, x_2 and x_7, x_3, we can assume that ux_3 is not an edge. Let $S = \{x_1, x_4, x_6\}$. If u has no neighbor on S then $ux_5x_1x_2x_6x_3x_4x_7u$ is a 3-cycle. If u has exactly one neighbor in S, then by symmetry between x_1 and x_6 we may assume that ux_6 is not an edge and thus $ux_5x_1x_2x_6x_3x_7u$ is a 3-cycle. So u has at least

two neighbors in S. If u has three neighbors in S, then u has exactly two neighbors on $\{x_1, x_2, x_3, x_6\}$, a contradiction. So u has exactly two neighbors in S. If ux_1 and ux_4 are edges, then $ux_5x_1x_2x_6x_3x_7u$ is a 3-cycle. So, by symmetry between x_1 and x_4, we may assume that the two neighbors of u in S are x_4 and x_6. So then $ux_5x_1x_2x_6x_3x_7u$ is a 3-cycle, a contradiction. So we may assume that ux_7, and by symmetry ux_4 are not an edge..

So, $N_H(u) \subseteq \{x_5, x_1, x_3, x_6\}$ and, since we already proved that u does not have exactly two neighbors in $\{x_1, x_2, x_3, x_6\}$, one of the outcome holds. This proves (1).

(2) $N_H(v) \in \{\{x_7\}, \{x_4, x_7\}, \{x_3, x_7\}, \{x_2, x_7\}, \{x_6, x_7\}, \{x_2, x_4, x_6, x_7\}\}$.

By obvious symmetries in H, the proof is the same as the proof of (1). This proves (2).

Note that by claims (1) and (2), $u \neq v$. In the rest of the proof we show that, whatever the neighborhoods of u and v are, one can find a 3-cycle in $H \cup Q_{uv}$ (recall that Q_{uv} denote a unimodal path linking u and v).

Suppose first that $d_H(u) \leq 2$ and $d_H(v) \leq 2$.
If $d_H(u) = d_H(v) = 1$, then $ux_5x_1x_2x_6x_3x_4x_7vQ_{vu}u$ is a 3-cycle. So we may assume that $d_H(u) = 2$ and thus, by (1), u has exactly one neighbors in $\{x_1, x_3, x_6\}$. Note that by (2), vx_1 is not en edge. If $d_H(v) = 1$, then $ux_5x_1x_2x_6x_3x_7vQ_{uv}u$ is a 3-cycle. Moreover, it is still a 3-cycle if vx_4 is an edge. So $d_H(v) = 2$ and vx_4 is not an edge. Similarly, if ux_1 is an edge, then $vx_7x_4x_3x_6x_2x_5uQ_{uv}v$ is a 3-cycle. So ux_1 is not an edge. Now, by (1) and (2), both u and v has exactly one neighbor in $\{x_2, x_3, x_6\}$ and thus $ux_5x_2x_6x_3x_7vQ_{uv}u$ is a 3-cycle, a contradiction.

So, from now on, we assume that $d_H(u)$ and $d_H(v)$ are not both inferior to 2. Hence we may assume w.l.o.g. that $d_H(u) > 2$, and thus, by (1), $N_H(u) = \{x_1, x_3, x_5, x_6\}$.

Recall that by (2), vx_1 is not an edge. If v has no neighbor in $\{x_2, x_3\}$, then $x_5x_1x_2x_3x_7vQ_{vu}u$ is a 3-cycle. So v has at least one neighbor in $\{x_1, x_2, x_3\}$ and by (2). If $N_H(v) \subseteq \{\{x_2, x_7\}, \{x_3, x_7\}\}$, then $ux_5x_2x_3x_4x_7vQ_{uv}u$ is a 3-cycle. So by (2), $N_H(v) = \{x_2, x_4, x_6, x_7\}$

If uv is not an edge, then $ux_5x_2vx_7x_4x_3u$ is a 3-cycle with chords x_2x_3, x_3x_7 and vx_4, a contradiction. So we may assume that uv is an edge. Let u' and v' be fathers of respectively u and v and note that, since u' and v' are in $S_{i-2}(z)$, they have no neighbors in H. Therefore $u'ux_5x_2x_3x_7vv'Q_{v'u'}u'$ is a 3-cycle, with chords uv, ux_3 and vx_2, a contradiction. $\square$

Lemma 5.8 *Let G be a (K_4, 3-cycle)-free graph and let z be a vertex of H. Then, for every integer i, $G[S_i(z)]$ is butterfly-free.*

PROOF — Assume by way of contradiction that there exists an integer i such that $G[S_i(z)]$ contains a butterfly as an induced subgraph. Name it H and refer to Figure 6 for the name of its vertices. Let u be a father of y_4 and v be a father of y_5.

The two next claims examine what are the possible neighborhoods of u and v in H.

(1) $N_H(u) \in \{\{y_4\}, \{y_2, y_4\}, \{y_1, y_3, y_4\}\}$

Assume first that $|N_H(u)| = 2$. If $N_H(u) = \{y_1, y_4\}$, then $uy_1y_2y_3y_5y_4u$ is a 3-cycle, a contradiction. If $N_H(u) = \{y_3, y_4\}$, then $uy_4y_1y_2y_5y_3u$ is a 3-cycle, a contradiction. If $N_H(u) = \{y_4, y_5\}$, then $uy_4y_1y_2y_3y_5u$ is a 3-cycle, a contradiction. So, if $|N_H(u)| = 2$, then $N_H(u) = \{y_2, y_4\}$ and one of the outcome of the theorem holds.

Assume now that $|N_H(u)| = 3$. If $N_H(u) = \{y_2, y_3, y_4\}$, then $uy_4y_2y_5y_3u$ is a 3-cycle, a contradiction. If $N_H(u) = \{y_1, y_3, y_4\}$, then one of the outcome of the theorem holds. So, since G is K_4-free, u has to see y_5. The third neighbor of u is thus y_1 or y_3 and, by symmetry, we may assume that it is y_1. Therefore $uy_4y_1y_2y_5u$ is a 3-cycle, a contradiction.

So we may assume that $|N_H(u)| \geq 4$. Since G is K_4-free, $|N_H(u)| = 4$ and $N_H(u) = \{y_1, y_3, y_4, y_5\}$. Thus $uy_1y_4y_2y_5u$ is a 3-cycle, a contradiction. This proves (1).

(2) $N_H(v) \in \{\{y_5\}, \{y_2, y_5\}, \{y_1, y_3, y_5\}\}$

By obvious symmetries in H, the proof is the same as the proof of (1). This proves (2).

Note that by claims (1) and (2), $u \neq v$. In the rest of the proof we show that, whatever the neighborhoods of u and v are, one can find a 3-cycle in $H \cup Q_{uv}$ (recall that Q_{uv} denote a unimodal path linking u and v).

Case 1 : $N_H(v) = \{y_5\}$.
If $N_H(u) = \{y_4\}$ then $uy_4y_1y_2y_3y_5vQ_{vu}u$ is a 3-cycle, a contradiction. So $N_H(u) \in \{\{y_2, y_4\}, \{y_1, y_3, y_4\}\}$ and thus $uy_4y_2y_3y_5vQ_{vu}u$ is a 3-cycle, a contradiction. This completes the proof in case 1.

So from now on, we may assume that $N_H(v) \neq \{y_5\}$ and, by symmetry, that $N_H(u) \neq \{y_4\}$.

Case 2 : $N_H(v) = \{y_2, y_5\}$.

If $N_H(u) = \{y_2, y_4\}$ then $uy_4y_2y_5vQ_{vu}u$ is a 3-cycle. Otherwise, we may assume that $N_H(u) = \{y_1, y_3, y_4\}$ and then $uy_1y_2y_3y_5vQ_{vu}$ is a 3-cycle, a contradiction.

So from now on, we may assume that $N_H(v) \neq \{y_2, y_5\}$ and by symmetry, $N_H(u) \neq \{y_2, y_4\}$. This leads to the following last case.

Case 3 : $N_H(v) = \{y_1, y_3, y_5\}$ and $N_H(u) = \{y_1, y_3, y_4\}$.
If uv is not an edge then $uy_1y_4y_5vy_3u$ is a 3-cycle, with chords uy_4, vy_1 and y_3y_5, a contradiction. So uv is an edge. Let u' and v' be fathers of respectively u and v. If u' (or v') is adjacent to both u and v we assume that $u' = v'$. Note that since u' and v' are in S_{k-2} they have no neighbors in H. Therefore $u'uy_1y_2y_3vv'Q_{u'v'}u'$ is a 3-cycle, with chords uv, uy_3 and vy_1, a contradiction. This completes the proof in Case 3.

□

We can now give the proof of Theorem 5.2 recall that it states that every (K_4,3-cycle)-free graph has chromatic number at most $4c$.

PROOF — Assume by contradiction that there exists a (K_4, 3-cycle)-free graph G that satisfies $\chi(G) \geq 4c + 1$ and let z be a vertex of G. By Remark 3.2, there exists an integer k such that $S_k(z, G)$ has chromatic number at least $2c+1$. So, by Lemma 5.6, it must contain a dragonfly or a butterfly, which is a contradiction with Lemma 5.7 or Lemma 5.8. □

5.3 Clique number at least 4 : proof of Theorem 5.3

Recall that Theorem 5.2 states that every (3-cycle)-free graph has chromatic number at most $\max(4c, \omega(G) + 1)$.

PROOF — Consider by contradiction the smallest (in number of vertices) graph $G \in \mathcal{C}_3$ such that $\chi(G) > \max(\omega(G)+1, 4c)$. By Theorem 5.1 and 5.2, we have $\omega(G) \geq 4$. Put $\omega(G) = \omega$. Let K be a largest clique of G and denote by $x_1, \ldots, x_\omega$ the vertices of K.

(1) Every vertex of G is of degree at least $\omega + 1$.

If a vertex v of G is of degree at most ω, then by minimality of G we can color $G \setminus \{v\}$ with $max(\omega(G) + 1, 4c)$ colors and extend the coloring to G, a contradiction. This proves (1).

(2) G does not admit clique cutsets.

Assume by contradiction that G has a clique cutset A. Let C_1 be a connected component of $G \setminus A$, and C_2 the union of all others components. By minimality of G, we may color $G[C_i \cup K]$ with $max(\omega(G)+1, 4c)$ colors for $i = 1, 2$. By using the same colors for the vertices of A in the coloring of $G[C_1 \cup K]$ and $G[C_2 \cup K]$, we can extend the coloring to G, a contradiction. This proves (2).

(3) If $u \in N(K)$, then $d_K(u) = 1$ or $\omega - 1$.

Assume by way of contradiction that u has at least two neighbors in K, say x_1 and x_2, and at least two non-neighbors, say x_3 and x_4. Then $ux_1x_3x_4x_2u$ is a 3-cycle, with chords x_1x_2, x_1x_4 and x_2x_3, a contradiction. This proves (3).

Define $S_i = \{u \in N(K) | N_K(u) = \{x_i\}\}$, $T_i = \{u \in N(K) | N_K(u) = V(K) \setminus \{x_i\}\}$ and, for all $i = 1, \ldots, \omega$, $U_i = S_i \cup T_i$.

An uv-path P is an *N(K)-connection* if no vertex of P is in K and $N(K) \cap P = \{u, v\}$. Note that vertices of $\mathring{P}$ have no neighbors on K and that an $N(K)$-connection can be an edge.

(4) Let P be an $N(K)$-connection with endvertices u and v. Then there exists an integer i such that $\{u, v\} \subseteq U_i$ and $\{u, v\} \not\subseteq T_i$.

Let i, j, k and l be 4 distinct integers in $\{1, \ldots, \omega\}$. Such integers exist since $\omega \geq 4$.
If $u \in T_i$ and $v \in T_j$, then $ux_jx_kx_ivPu$ is a 3-cycle, with chords ux_k, vx_k and x_ix_j.
If $u \in S_i$ and $v \in T_j$, then $ux_ix_kx_lvPu$ is a 3-cycle, with chords vx_i, vx_k and x_ix_l.
If $u \in S_i$ and $v \in S_j$, then $ux_ix_kx_lx_jvPu$ is a 3-cycle, with chords x_ix_j, x_ix_l and x_jx_k.
If $u \in T_i$ and $v \in T_i$, then $ux_jx_ix_kvPu$ is a 3-cycle, with chords ux_j, vx_k and x_jx_k. This proves (4).

(5) There is a unique $i \in \{1, \ldots, \omega\}$ for which $U_i \neq \emptyset$.

Let us argue by way of contradiction. By (2), $G \setminus K$ is connected, so there exists a path P in $G \setminus K$ from U_i to U_j such that $i \neq j$. Choose P subject to its minimality. It is clear that P is an $N(K)$-connection and thus it contradicts (4). This proves (5).

By (5), we may assume w.l.o.g. that $U_1 \neq \emptyset$ and, for any $i \neq 1$, $U_i = \emptyset$. Moreover, S_1 and T_1 both contain at least two vertices, otherwise x_1 or x_2 have degree at most ω, a contradiction to (1).

We say that a vertex x is *complete* to a set of vertex S is x is adjacent to every vertex in S.

(6) *If there exists an $N(K)$-connection from a vertex of T_1 to a vertex $s_1 \in S_1$, then s_1 is complete to T_1.*

Let P be a minimal $N(K)$-connection from s_1 to T_1. Denote by $t_1 \in T_1$ the second endvertex of P. Assume by way of contradiction that there exists a vertex $t_2 \in T_1 \setminus \{t_1\}$ that is not adjacent to s_1. Then there is no edge linking t_2 with a vertex of P, otherwise there would be an $N(K)$-connection from t_1 to t_2, contradicting (4). So, $s_1Pt_1x_2t_2x_3x_1s_1$ is a 3-cycle with chords x_1x_2, t_1x_3 and x_2x_3, a contradiction.

So s_1 is complete to $T_1 \setminus \{t_1\}$ and, by minimality of P, s_1 is adjacent to t_1. This proves (6).

(7) *$N(T_1) \subseteq S_1 \cup K$.*

Assume by contradiction that there exists $t_1 \in T_1$ such that $N(t_1) \nsubseteq S_1 \cup K$. Since t_1 is not a cutvertex by (2), consider a minimal path P' from $N(t_1) \setminus (S_1 \cup K)$ to $N(K)$ in $G \setminus \{t_1\}$. Call t_1' and x the extremities of P with $t_1' \in N(t_1) \setminus (S_1 \cup K)$ and put $P = t_1Px$. Observe that if t_1x is not an edge, then t_1Px is an $N(K)$-connection and that in both cases there exists an $N(K)$-connection linking t_1 and x. So, by (4), $x \notin T_1$. Hence, by (6), x is complete to T_1 and in particular xt_1 is an edge. Finally $t_1Ps_1x_1x_2x_3t_1$ is a 3-cycle with chords s_1t_1, x_1x_3 and t_1x_2, a contradiction. This proves (7).

(8) *For any vertex $t \in T_1$, $N(t) = S_1 \cup K \setminus \{x_1\}$.*

Let $t \in T_1$. By (4), T_1 is a stable set. So if t is not adjacent to any vertex of S_1, $N(t) = K \setminus \{x_1\}$, a contradiction to (1). So t is adjacent to at least one vertex in S_1 and thus, by (6), t is complete to S_1. This proves (8).

Let t_1 and t_2 be two distinct vertices in T_1 (remind that they exist because if $|T_1| = 1$, then $d(x_2) = \omega$, contradicting (1)). By (8), $N(t_1) = N(t_2) = S_1 \cup K \setminus \{x_1\}$. By minimality of G, $G \setminus \{t_2\}$ admits a proper coloring γ with $\max(4c, \omega + 1)$ colors. Since $t_1t_2 \notin E(G)$ and $N(t_1) = N(t_2)$, γ can be extended to G by giving to t_2 the same color as t_1. □

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